Exercises from Atiyah-MacDonald
Introduction to Commutative Algebra

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Chapter 1

Throughout, a ring is assumed to be commutative and with unity.

Proposition 0.1 (Exercise 1)**.** *Let A be a ring, and let $x \in A$ be nilpotent and $u \in A$ be a unit. Then $x + u$ is a unit, with inverse given by*

$$(x+u)^{-1} = u^{-1} \sum_{k=0}^{\infty} (-xu^{-1})^k$$

Proof. We may assume $x \neq 0$, since in that case $x + u = u$ is a unit. First, consider the case $u = 1$. As x is nilpotent, choose n so that $x^n = 0$. Define

$$y = \sum_{k=0}^{n-1} (-x)^k = 1 - x + x^2 - x^3 + \ldots + (-x)^{n-1}$$

Then we compute

$$\begin{aligned} (x+1)y &= \big(x - x^2 + x^3 + \ldots + (-1)^{n-1}x^n\big) + \big(1 - x + x^2 + \ldots + (-x)^{n-1}\big) \\ &= 1 + (-1)^{n-1}x^n = 1 \end{aligned}$$

Thus $x + 1$ is a unit. Now let $u \in A$ be any unit. Then xu^{-1} is nilpotent, so by the previous case, $xu^{-1} + 1$ is a unit. Writing

$$x + u = u(xu^{-1} + 1)$$

We see that $x + u$ is a unit as well. Concretely, the inverse is

$$u^{-1} \sum_{k=0}^{\infty} (-xu^{-1})^k$$

Note that only finitely many terms are nonzero because xu^{-1} is nilpotent. □

Proposition 0.2 (Exercise 5)**.** *Let A be a ring and let $A[[x]]$ be the ring of formal power series with coefficients in A. Let $\iota : A \hookrightarrow A[[x]]$ be the inclusion.*

(a) $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$ is a unit if and only if a_0 is a unit in A.

(b) If $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$ is nilpotent, then a_n is nilpotent for all $n \geq 0$.

(c) $f = \sum_{n=0}^{\infty} a_n x^n$ belongs to the Jacobson radical of $A[[x]]$ if and only if a_0 belongs to the Jacobson radical of A.

(d) The contraction $\mathfrak{m}^c = \iota^{-1}(\mathfrak{m})$ of a maximal ideal $\mathfrak{m} \subset A[[x]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c = \iota^{-1}(\mathfrak{m})$ and x.

(e) Every prime ideal of A is the contraction of a prime ideal in $A[[x]]$.

Proof. (a) Suppose $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$ is a unit, with inverse $f^{-1} = \sum_{n=0}^{\infty} b_n x^n$. Then

$$1 = (a_0 + a_1 x + \ldots)(b_0 + b_1 x + \ldots) = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + \ldots$$

So all the nonconstant terms vanish, and $a_0 b_0 = 1$, hence a_0 is a unit.

Conversely, suppose a_0 is a unit. By mulitplying by a_0^{-1}, we reduce to showing that a power series of the form $1 + xf$ is a unit, where $f \in A[[x]]$. We can write down the inverse to $1 + f$ explicitly as

$$(1 + xf)^{-1} = \sum_{n=0}^{\infty} (-xf)^n = 1 - xf + x^2 f^2 - x^3 f^3 + \ldots$$

Note that this series makes sense in $A[[x]]$, because we can recursively compute the low degree terms. The coefficient for the x^n term of the resulting power series is determined by the first n (finitely many) terms $1, -xf, x^2 f^2, \ldots x^n f^n$, so this power series only requires finite sums in A.

(b) Suppose $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$ is nilpotent, with $f^m = 0$ for some m. We will prove that all the a_i are nilpotent by induction on i. First, since the constant term of f^m is a_0^m and $f^m = 0$, $a_0^m = 0$, so a_0 is nilpotent. For the inductive step, suppose $a_0, \ldots a_{i-1}$ are nilpotent. Since the nilpotent elements form a subring, the element

$$f - \left(a_0 + a_1 x + a_2 x^2 + \ldots + a_{i-1} x^{i-1}\right) = x^i \left(a_i + a_{i+1} x + a_{i+2} x^2 + \ldots\right)$$

is nilpotent. Since x^i is not nilpotent, the other factor must be, so by the base case, a_i is nilpotent. This completes the induction.

(c) We denote the Jacobson radical of A by $J(A)$. Let $f = \sum a_n x^n \in A[[x]]$, and suppose $a_0 \in J(A)$. By Proposition 1.9 of Atiyah-MacDonald, $1 - a_0 b$ is a unit in A for all $b \in A$.

Let $g = \sum b_n x^n \in A[[x]]$. The constant term of $1 - fg$ is $1 - a_0 b_0$, which is a unit in A, so by part (b), $1 - fg$ is a unit in $A[[x]]$. Applying Proposition 1.9 of Atiyah-MacDonald again, $f \in J\left(A[[x]]\right)$.

The converse is essentially the same argument. Suppose $f \in J\left(A[[x]]\right)$. Then $1 - fg$ is a unit in $A[[x]]$ for all $g = \sum b_n x^n \in A[[x]]$, so by part (b) the constant term $1 - a_0 b_0$ is a unit in A. We can have g with any constant term b_0, so $1 - a_0 b$ is a unit in A for all $b \in A$, hence by Proposition 1.9, $a_0 \in J(A)$.

(d) Let $\mathfrak{m} \subset A[[x]]$ be a maximal ideal. Suppose $\iota^{-1}(\mathfrak{m}) \subset A$ is not maximal, so there is an element $a \in A$ so that $\iota^{-1}(\mathfrak{m}) \subsetneq \iota^{-1}(\mathfrak{m}) + (a) \subsetneq A$. In particular, $a \notin \iota^{-1}(\mathfrak{m})$, so $\iota(a) = a \notin \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, $\mathfrak{m} + (a) = A$, so there exist $m \in \mathfrak{m}, b \in A$ so that

$$1 = m + ba \qquad \text{(equality in } A[[x]])$$

Since $1, ba \in A$, we get $m \in A$, so $m \in \iota^{-1}(\mathfrak{m})$. Thus $\iota^{-1}(m) + (a) = A$, which is a contradiction. Thus $\iota^{-1}(\mathfrak{m})$ is maximal.

Now we show that $\mathfrak{m} \subset A[[x]]$ is generated by $\iota^{-1}(\mathfrak{m})$ and x (as an ideal of $A[[x]]$). Since $0 \in J(A)$, by part (c) $x \in J(A[[x]])$, so $x \in \mathfrak{m}$. Now note that $\iota^{-1}(\mathfrak{m}) = \mathfrak{m} - (x)$, so $\mathfrak{m} = \iota^{-1}(\mathfrak{m}) + (x)$, which is to say, $\mathfrak{m}$ is generated by $\iota^{-1}(\mathfrak{m})$ and x.

(e) Consider following commutative diagram.

$$\begin{array}{ccc} A & \hookrightarrow & A[[x]] \\ & \overset{\cong}{\searrow} & \downarrow \\ & & A[[x]]/(x) \end{array}$$

where $A \to A[[x]]$ is the natural inclusion, $A[[x]] \to A[[x]]/(x)$ is the canonical quotient map, and $A \to A[[x]]/(x)$ is the isomorphism $a \mapsto \overline{a}$. By Proposition 1.1 of Atiyah-MacDonald, $\mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p})$ gives a bijection between prime ideals of $A[[x]]/(x)$ and prime ideals of $A[[x]]$. Via the isomorphism $A \cong A[[x]]/(x)$ in our commutative triangle, prime ideals of A correspond to prime ideals of $A[[x]]$ that contain (x). That is to say, every prime ideal of A is the contraction of a prime ideal of $A[[x]]$. □

Lemma 0.3 (for Exercise 7)**.** *Let A be a ring such that for every $x \in A$, there exists $a \in A$ so that $x = x^2 a$. Then every prime ideal of A is maximal.*

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal. We will show that $\mathfrak{p}$ is maximal by showing that $A/\mathfrak{p}$ is a field. Let $\overline{x} \in A/\mathfrak{p}$ be nonzero, and choose a representative $x \in A$, so $x \neq 0$. The relation $x = x^2 a$ in A gives $\overline{x} = \overline{x}^2 \overline{a}$ in $A/\mathfrak{p}$, which we write as

$$\overline{x}^2 \overline{a} - \overline{x} = \overline{x}\left(\overline{xa} - \overline{1}\right) = 0$$

Since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is an integral domain, so one of the factors is zero. Since $\overline{x} \neq 0$ by assumption, $\overline{xa} = \overline{1}$, thus $\overline{x}$ is invertible in $A/\mathfrak{p}$ with inverse $\overline{a}$. □

Corollary 0.4 (Exercise 7)**.** *Let A be a ring so that every $x \in A$ satisfies $x^n = x$ for some $n > 1$ (depending on x). Then every prime ideal of A is maximal.*

Proof. Apply the previous lemma with $a = x^{n-2}$. □

Exercise 16.
Draw pictures of $\operatorname{spec}\mathbb{Z}, \operatorname{spec}\mathbb{R}, \operatorname{spec}\mathbb{C}[x], \operatorname{spec}\mathbb{R}[x]$. **Solution.**

We begin with some analysis applicable to all rings A. Unless A is a field, the point $\{(0)\} \in \operatorname{spec} A$ is not in any basic closed set, so (0) is contained in every open set, and the closure of (0) is $\operatorname{spec} A$. More generally, a singleton set $\{\mathfrak{p}\} \subset \operatorname{spec} A$ is closed if and only if $\mathfrak{p}$ is a maximal ideal.

As a set, $\operatorname{spec}\mathbb{Z}$ is

$$\operatorname{spec}\mathbb{Z} = \{(0)\} \cup \{(p) : p \text{ prime}\}$$

The basic closed sets are

$$V(E) = \{(p) : n \in (p), \forall n \in E\} = \{(p) : p|n, \forall n \in E\} = \{(p) : p| \gcd(E)\}$$

where $E \subset \mathbb{Z}$. Note that $V(E)$ is finite. Thus the basic open sets are complements these finite subsets of $\operatorname{spec}\mathbb{Z}\setminus(0)$, and every open set contains the point (0). The subspace topology on $\operatorname{spec}\mathbb{Z} \setminus (0)$ is the finite complement topology - all open sets are complements of finite sets. We depict this as below, using a different marker for the point (0) as a reminder that this point is different.

$*$	$\bullet$	$\bullet$	$\bullet$	$\dots$
(0)	(2)	(3)	(4)	$\dots$

Since $\mathbb{R}$ is a field, the only ideals are (0) and $\mathbb{R}$, so the only prime ideal is (0). So as a set, $\operatorname{spec}\mathbb{R} = \{(0)\}$. Since (0) is the only point, it is a closed and open set. We depict $\operatorname{spec}\mathbb{R}$ as

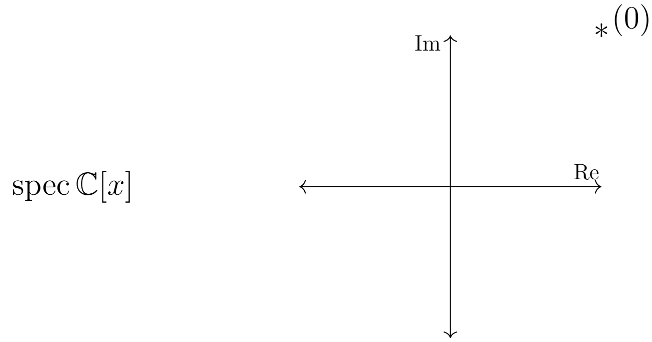
$\bullet$
(0)

$\mathbb{C}$ is algebraically closed, so the irreducible elements of $\mathbb{C}[x]$ are linear polynomials. $\mathbb{C}[x]$ is a PID, so the prime ideals are all principal ideals $(x-a)$ for $a \in \mathbb{C}$. There is also the zero ideal (0). Note that $(x-0)$ is not the same as (0). So, as a set, we can think of $\mathbb{C}[x]$ as $\mathbb{C} \cup \{(0)\}$, with the point $a \in \mathbb{C}$ corresponding to the ideal $(x-a)$. The basic closed sets are

$$\begin{aligned} V(E) &= \{(x-a) : f \in (x-a), \forall f \in E\} \\ &= \{(x-a) : (x-a)|f, \forall f \in E\} \\ &= \{(x-a) : f(a) = 0, \forall f \in E\} \end{aligned}$$

so the basic closed sets are common zero loci of sets of polynomials. The zero locus of a family of polynomials is the same as the zero locus of the ideal generated by that family, and $\mathbb{C}[x]$ is Noetherian, so we can always reduce to a finite family of polynomials. A polynomial has only finitely many roots, so the basic closed sets are finite sets.

Identifying $\operatorname{spec}\mathbb{C}[x]$ with $\mathbb{C} \cup \{(0)\}$ via $(x-a) \leftrightarrow a$, we depict $\operatorname{spec}\mathbb{C}[x]$ as the complex plan, except that the topology is the finite complement topology, and there is an extra point (0), whose closure is the entire space.



$\mathbb{R}[x]$ is a PID, but $\mathbb{R}$ is not algebraically closed, so there are irreducible nonlinear polynomials. However, the irreducible polynomials are either linear or quadratic. An irreducible quadratic must be monic, and the two roots are complex conjugates. Thus the irreducible quadratics have the form

$$(x - z)(x - \overline{z}) = x^2 - 2\operatorname{Re}(z)x + |z|^2$$

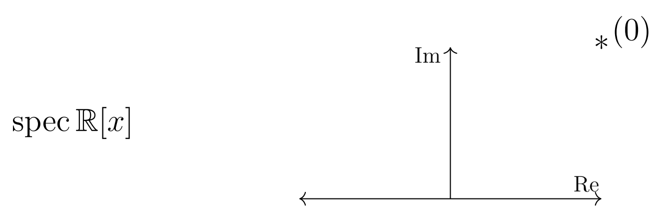
We can parametrize these by the upper half-plane of $\mathbb{C}$.

$$\mathbb{R}[x] = \{(0)\} \cup \{(x - a) : a \in \mathbb{R}\} \cup \left\{(x^2 - 2\operatorname{Re}(z) + |z|^2) : z \in \mathbb{C}, \operatorname{Im}(z) > 0\right\}$$

We identify $(x - a)$ with $a \in \mathbb{R}$, and $(x^2 - 2\operatorname{Re}(z) + |z|^2)$ with $z \in \mathbb{C}, \operatorname{Im}(z) > 0$. The basic closed sets are

$$\begin{aligned} V(E) &= \{(g) : f \in (g), \forall f \in E\} \\ &= \{(g) : g | f, \forall f \in E\} \\ &= \{(x - a) : f(a) = 0, \forall f \in E\} \cup \left\{(x^2 - 2\operatorname{Re}(z) + |z|^2) : f(z) = f(\overline{z}) = 0, \forall f \in E\right\} \end{aligned}$$

so as with $\mathbb{C}[x]$, we think of $V(E)$ as the common zero locus of the polynomials $f \in E$. Since $\mathbb{R}[x]$ is Noetherian, we need only consider finite sets of polynomials, which have a finite number of zeroes each, so the basic closed sets are finite subsets. Using our identification, we think of $\operatorname{spec}\mathbb{R}[x]$ as the real line, union with the open upper half plane $\operatorname{Im}(z) > 0$, along with a single "fuzzy" point (0) whose closure is the whole space.



Proposition 0.5 (Exercise 17)**.** *Let A be a ring and let $X = \operatorname{spec} A$. For $f \in A$, set*

$$X_f = X \setminus V(f) = \{p \in X : f \notin p\}$$

Then

1. *The sets X_f form a basis of open sets for the Zariski topology on* $\operatorname{spec} A$.
2. $X_f \cap X_g = X_{fg}$.

3. $X_f = \emptyset \iff f$ *is nilpotent.*

4. $X_f = X \iff f$ *is a unit.*

5. $X_f = X_g \iff \sqrt{(f)} = \sqrt{(g)}$ *($\sqrt{\cdot}$ denotes the radical of an ideal).*

6. X *is quasi-compact. (Every open cover of* X *has a finite subcover.)*

7. Each X_f *is quasi-compact.*

8. An open subset of X *is quasi-compact if and only if it is a finite union of sets* X_f*.*

Proof. (1) Recall that the Zariski topology is defined by taking the sets $V(E) = \{p \in X : E \subset p\}$ as closed. Let $U \subset X$ be open. Then $U = X \setminus V(E)$ for some $E \subset A$. Then

$$U = \{q \in X : E \not\subset q\} = \{q \in X : \exists f \in E, f \notin q\}$$

so for $f \in E$, $X_f \subset U$, and hence

$$\bigcup_{f \in E} X_f \subset U$$

On the other hand, for $p \in U$, choose $f \in E$ with $f \notin p$, and then $p \in X_f$, so $U \subset \bigcup X_f$. Thus $U = \bigcup X_f$, so the x_f form a basis of open sets for X.

(2) If $p \in X_f \cap X_g$, then $f, g \notin p$. Since p is prime, $fg \notin p$, so $p \in X_{fg}$, thus $X_f \cap X_g \subset X_{fg}$. For the reverse inclusion, if $p \in X_{fg}$, then $fg \notin p$. If $f \in p$, then $fg \in p$ since p is an ideal, so $f \notin p$. Similarly, $g \notin p$, so $p \in X_f \cap X_g$, thus $X_{fg} \subset X_f \cap X_g$.

(3) If f is nilpotent, with $f^n = 0$, and $p \in X$ is a prime ideal of A, then since $0 \in p$, either $f \in p$ or $f^{n-1} \in p$. Continuing like this, we conclude that $f \in p$. Thus $X_f = \emptyset$. Conversely, if $X_f = \emptyset$, then every prime ideal of A contains f, so f is nilpotent by Proposition 1.8 of Atiyah-MacDonald.

(4) If f is a unit, the any ideal containing f is A, so no prime ideals contain f, so $X_f = X$. Conversely, if $X_f = X$, then no prime ideals contain f. Then no maximal ideals contain f. By Corollary 1.3 of Atiyah-MacDonald, every non-unit is contained in a maximal ideal, so f must be a unit.

(5) Suppose $\sqrt{(f)} = \sqrt{(g)}$. By symmetry, it is sufficient to show $X_f \subset X_g$. Let $p \in X_f$. Since $\sqrt{(f)} = \sqrt{(g)}$, in particular $f \in \sqrt{(g)}$, so there exists $n \in \mathbb{Z}, a \in A$ with $f^n = ga$. Since $f \notin p$ and p is prime, $f^n \notin p$, so $ga \notin p$. Then $g \notin p$, so $p \in X_g$. Thus $X_f \subset X_g$.

Conversely, suppose $X_f = X_g$. By Proposition 1.14 of Atiyah-MacDonald, $\sqrt{(f)}$ is equal to the intersection of all prime ideals containing (f), which is to say,

$$\sqrt{(f)} = \bigcap_{f \in p} p = \bigcap_{p \notin X_f} p \qquad \sqrt{(g)} = \bigcap_{g \in p} p = \bigcap_{p \notin X_g} p$$

Since $X_f = X_g$, these two intersections are over the same set of prime ideals in $X = \operatorname{spec} A$, so they are the same, thus $\sqrt{(f)} = \sqrt{(g)}$.

(6) Given an arbitrary open cover of X, we can cover each of the open sets with sets of type X_f, so we assume our open cover of X is of sets X_{f_i}, $i \in I$.

$$X = \bigcup_{i \in I} X_{f_i}$$

This says that for every $p \in X$, there exists $i \in I$ so that $f_i \notin p$. That is, the set $\{f_i\}_{i \in I}$ has non-empty intersection with the complement of every prime (or maximal) ideal. Thus the same is true for the ideal generated by all f_i, so that ideal is not contained in any maximal ideal, so it is all of A. Thus we can write $1 \in A$ as a (finite) linear combination of the f_i, so there is a finite subset $J \subset I$ and elements $a_i \in A$ for $i \in J$ so that

$$1 = \sum_{i \in J} a_i f_i$$

Now we claim that

$$X = \bigcup_{i \in J} X_{f_i}$$

Let $p \in X$. Then there exists $b \in A \setminus p$, and we can write b as

$$b = \sum_{i \in J} b a_i f_i$$

If all f_i for $i \in J$ were in p, then $b \in p$, so some f_i is not in p. Thus $p \in X_{f_i}$, so X is covered by finite subcover X_{f_i} for $i \in J$.

(7) and (8) I don't know how to prove these. □

Proposition 0.6 (Exercise 18). *Let A be a ring and $X = \operatorname{spec} A$, and let $p \in X$.*

1. *$\{p\} \subset X$ is closed if and only if p is a maximal ideal.*
2. *$\overline{\{p\}} = V(p)$ (the overline denotes the closure).*
3. *$q \in \overline{\{p\}} \iff p \subset q$.*
4. *X is a T_0-space. (For $p, q \in X$ distinct, there is a neighborhood of p not containing q, or there is a neighborhood of q not containing p.)*

Proof. (1) Let p be a maximal ideal. Then $V(p) = \{q \in X : p \subset q\} = \{p\}$ is closed.

Conversely, let p be a prime ideal so that $\{p\}$ is closed. Suppose p is a properly contained in a maximal ideal m. Since $\{p\}$ is closed and $m \in X \setminus \{p\}$ and $X \setminus \{p\}$ is open, there exists $f \in A$ so that $m \in X_f$ and $p \notin X_f$. Then $f \in p \setminus m$, which contradicts $p \subset m$. Thus p is maximal.

(2) Recall that the closure of a set is the intersection of all closed sets containing it, so

$$\overline{\{p\}} = \bigcap_{\{p\} \subset V} V = \bigcap_{p \in V} V = \bigcap_{E \subset p} V(E)$$

thus $\overline{\{p\}} \subset V(p)$. For the reverse inclusion, let $q \in V(p)$, so $p \subset q$. Then $E \subset p \implies E \subset q$, so $q \in \bigcap_{E \subset p} V(E) = \overline{\{p\}}$.

(3) By part (2), $\overline{\{p\}} = V(p) = \{q \in X : p \subset q\}$.

(4) Let $p, q \in X$ be distinct. Then at least one of p, q is not a subset of the other, say p is not a subset of q without loss of generality. Then there exists $f \in q \setminus p$, so $p \in X_f$ and $q \notin X_f$, so X_f is the required neighborhood of p. □

Definition 0.1. A topological space is **irreducible** if $X \neq \emptyset$ and if every pair of nonempty open sets in X have nonempty intersection, or equivalently if every nonempty open subset of X is dense in X.

Proposition 0.7 (Exercise 19)**.** *Let A be a ring and $X = \operatorname{spec} A$. X is irreducible if and only if the nilradical of A is a prime ideal. (Even stronger, if the nilradical of A is a prime ideal, the intersection of any two nonempty open subsets of X contains the nilradical.)*

Proof. Let N be the nilradical of A, and suppose N is prime. By Proposition 1.8 of Atiyah-MacDonald, N is contained in every prime ideal of A. To show that X is irreducible, it suffices to show that any two nonempty basic open sets X_f, X_g have nonempty intersection.

If X_f is a nonempty basic open set, there is a prime p with $f \notin p$, so $f \notin N$, so $N \in X_f$. Thus N is contained in every basic open subset, so the intersection of any two basic open subsets X_f, X_g is nonempty. Thus X is irreducible. (Since every open subset of X contains some X_f.

Conversely, suppose X is irreducible. To show that N is prime, we show that for $f, g \notin N$, the product fg is not in N. Let $f, g \in A \setminus N$. Since X is irreducible, $X_f \cap X_g = X_{fg}$ is nonempty, so there is a prime $\mathfrak{p}$ so that $\mathfrak{p} \in X_{fg}$, or equivalently, $fg \notin \mathfrak{p}$. Since $N \subset \mathfrak{p}$, we get $fg \notin N$. □

Chapter 3

Proposition 0.8 (Exercise 1)**.** *Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Then $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.*

Proof. If there exists $s \in S$ with $sM = 0$, then for $\frac{m}{t} \in S^{-1}M$, we have $\frac{m}{t} = \frac{0}{1}$ since $s(m1 - 0t) = sm = 0$. For the converse, suppose that $S^{-1}M = 0$. Choose a set of generators $x_1, \dots, x_n$ for M. By assumption, $\frac{x_i}{1} = \frac{0}{1}$ in $S^{-1}M$, so there exists $s_1, \dots, s_n \in S$ so that $s_i x_i = 0$. Then set $s = s_1 \dots s_n$. Then $sx_i = 0$ for each i, so $sM = 0$. □

Proposition 0.9 (Exercise 5)**.** *If A is a ring such that for every prime ideal p, the local ring A_p has no nonzero nilpotent elements, then A has no nonzero nilpotent elements.*

Proof. Let $a \in A$ be nilpotent, with $a^n = 0$. Then $\left(\frac{a}{1}\right)^n = \left(\frac{a^n}{1}\right) = \frac{0}{1} = 0$ in A_p for every prime ideal p, so $\frac{a}{1}$ is nilpotent in A_p, so $\frac{a}{1} = 0$ in A_p. Thus there exists $t_p \in A \setminus p$ so that $t_p a = 0$ (notably, $t_p \neq 0$).

That is to say, the annihilator $\operatorname{Ann}(a)$ is an ideal of A which has nonzero intersection with the complement of every prime ideal. By Corollary 1.4 of Atiyah-MacDonald, every proper ideal is contained in some maximal (and prime) ideal, so we conclude that $\operatorname{Ann}(a) = A$. Thus $1a = a = 0$, so zero is the only nilpotent of A. □

Remark: As can be seen from the proof of the previous proposition, the hypothesis may be weakened to get a stronger statement: If A is a ring such that for every maximal ideal p, the local ring A_p has no nonzero nilpotent elements, then A has no nonzero nilpotent elements.

Remark: If every localization A_p is an integral domain, it does not follow that A is an integral domain. It is not even sufficient to have each localization be a field. We give a counterexample.

Let $n \in \mathbb{Z}$ be a product of distinct primes, $n = p_1 \dots p_k$ with $k \geq 2$. By the ideal correspondence between $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, all prime ideals of $\mathbb{Z}/n\mathbb{Z}$ are principal, generated by some $\overline{p}_i \in \mathbb{Z}/n\mathbb{Z}$. The localization of $\mathbb{Z}/n\mathbb{Z}$ at $(\overline{p}_i)$ is isomorphic to $\mathbb{Z}/p_i\mathbb{Z}$, which is a field, and hence an integral domain. However, $\mathbb{Z}/n\mathbb{Z}$ is not an integral domain.

Proposition 0.10 (Exercise 21(i))**.** *Let A be a ring and $S \subset A$ a multiplicative subset. Let $\phi : A \to S^{-1}A$ be the canonical homomorphism. Then $\phi^* : \operatorname{spec}(S^{-1}A) \to \operatorname{spec} A$ is a homeomorphism of $\operatorname{spec}(S^{-1}A)$ onto its image in $\operatorname{spec} A$. (In particular, for $f \in A$, the image of $\operatorname{spec} A_f$ is X_f, so $\operatorname{spec} A_f \cong X_f$.)*

Proof. By the ideal correspondence (Proposition 3.11 of Atiyah-MacDonald), extension and contraction give a bijection between prime ideals of $S^{-1}A$ and prime ideals of A that do not intersect S, so ϕ^* is injective, and $\operatorname{im} \phi^* = \{p \in \operatorname{spec} A : S \cap p = \emptyset\}$. It is sufficient to show that ϕ^* is a continuous and open map.

To show that ϕ^* is continuous, it suffices to consider the preimage of the intersection of $\operatorname{im} \phi^*$ with a basic open subset $X_f \subset \operatorname{spec} A$.

$$(\phi^*)^{-1} (X_f \cap \operatorname{im} \phi^*) = \{p^e \mid p \in \operatorname{spec} A, f \notin p, p \cap S = \emptyset\}$$

To show that this is open in $\operatorname{spec}(S^{-1}A)$, we need to show that each p^e is contained in a basic open subset $X_{\frac{a}{s}}$. We claim that $p^e \in X_{\frac{f}{1}}$. If not, then $\frac{f}{1} \in p^e$, so we can write $\frac{f}{1}$ as a linear combination

$$\frac{f}{1} = \sum_i \frac{a_i}{s_i} x_i$$

where $a_i \in A, s_i \in S, x_i \in p$. Finding a common denominator, we can rewrite this as

$$\frac{f}{1} = \frac{\sum b_i x_i}{s}$$

where $b_i \in A, s \in S$. Then there exists $t \in S$ so that

$$t\left(fs - \sum_i b_i x_i\right) = 0 \qquad \text{or equivalently} \qquad tfs = t \sum_i b_i x_i$$

In the last equality, the right hand side is in p, since $x_i \in p$. In the left hand side, $t, f, s \notin p$, so by primality of p, $tfs \notin p$, which is a contradiction. Thus $\frac{f}{1} \notin p^e$, so $p^e \in X_{\frac{f}{1}}$.

Now we show that ϕ^* is open. It is sufficient to show that for a basic open subset $X_{\frac{a}{s}} \subset \operatorname{spec}(S^{-1}A)$, the image $\phi^*(X_{\frac{a}{s}})$ is open in $\operatorname{spec} A$.

$$\phi^*(X_{\frac{a}{s}}) = \left\{p^c \mid p \in \operatorname{spec}(S^{-1}A), \frac{a}{s} \notin p\right\}$$

To show that this is open in $\operatorname{spec} A$, we need to show that each p^c is contained in a basic open subset $X_f \subset \operatorname{spec} A$ for some $f \in A$. Since $\frac{a}{s} \notin p$ and p is an ideal, $\frac{a}{1} \notin p$, so $a \notin p^c$. Thus $p_c \in X_a$. □

Definition 0.2. Let A, ϕ, S be as above, and let $X = \operatorname{spec} A$. We denote the image of ϕ^* by $\boldsymbol{S^{-1}X}$.

Proposition 0.11 (Exercise 21(ii))**.** *Let $f : A \to B$ be a ring homomorphism. Let $X = \operatorname{spec} A$ and $Y = \operatorname{spec} B$, and $f^* : Y \to X$ be the induced map $p \mapsto f^{-1}(p)$. Identifying $\operatorname{spec}(S^{-1}A)$ with its image $S^{-1}X$ in X, and $\operatorname{spec}(S^{-1}B) = \operatorname{spec}(f(S)^{-1}B)$ with its image $S^{-1}Y$ in Y, the induced map $(S^{-1}f)^* : \operatorname{spec}(S^{-1}B) \to \operatorname{spec}(S^{-1}A)$ is the restriction of f^* to $S^{-1}Y$, and $S^{-1}Y = (f^*)^{-1}(S^{-1}X)$.*

In other words, if $\phi : A \to S^{-1}A$ and $\psi : B \to S^{-1}B$ are the canonical homomorphisms, the following diagram commutes.

$$\begin{array}{ccc}
\operatorname{spec}(S^{-1}B) & \xrightarrow{(S^{-1}f)^*} & \operatorname{spec}(S^{-1}A) \\
\cong\big\downarrow\psi^* & & \cong\big\downarrow\phi^* \\
S^{-1}Y = \operatorname{im}\psi^* & & S^{-1}X = \operatorname{im}\phi^* \\
\big\downarrow & & \big\downarrow \\
Y = \operatorname{spec} B & \xrightarrow{f^*} & X = \operatorname{spec} A
\end{array}$$

Proof. Let $p \in \operatorname{spec}(S^{-1}B)$. Going around the top of the rectangle, we obtain

$$\phi^*(S^{-1}f)^*(p) = \phi^{-1}\Big((S^{-1}f)^{-1}(p)\Big) = \phi^{-1}\left\{\frac{a}{s} \in S^{-1}A \;\middle|\; \frac{f(a)}{s} \in p\right\} = \left\{a \in A \;\middle|\; \frac{f(a)}{1} \in p\right\}$$

Going around the bottom of the rectangle, we obtain

$$f^*\psi^*(p) = f^{-1}\Big(\psi^{-1}(p)\Big) = \left\{a \in A \;\middle|\; \frac{f(a)}{1} \in p\right\}$$

so the images are equal as subsets of A, so they are equal ideals in $\operatorname{spec} A$. $\square$

Proposition 0.12 (Exercise 21(iii))**.** *Let $f : A \to B$ be a ring homomorphism. Let $\mathfrak{a} \subset A$ be an ideal, and $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B. Let $\overline{f} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the induced homomorphism. If $\operatorname{spec}(A/\mathfrak{a})$ is identified with its canonical image $V(\mathfrak{a}) \subset \operatorname{spec} A$ and $\operatorname{spec}(B/\mathfrak{b})$ is identified with its image $V(\mathfrak{b}) \subset \operatorname{spec} B$, then $\left(\overline{f}\right)^*$ is the restriction of f^* to $V(\mathfrak{b})$.*

In other words, if $\pi_A : A \to A/\mathfrak{a}$ and $\pi_B : B \to B/\mathfrak{b}$ are the canonical projections, the following diagram commutes.

$$\begin{array}{ccc}
\operatorname{spec}(B/\mathfrak{b}) & \xrightarrow{(\overline{f})^*} & \operatorname{spec}(A/\mathfrak{a}) \\
\cong\big\downarrow\pi_B^* & & \cong\big\downarrow\pi_A^* \\
V(\mathfrak{b}) & & V(\mathfrak{a}) \\
\big\downarrow & & \big\downarrow \\
\operatorname{spec} B & \xrightarrow{f^*} & \operatorname{spec} A
\end{array}$$

Proof. Let $p \in \operatorname{spec}(B/\mathfrak{b})$. Going around the top of the rectangle, we obtain

$$\pi_A^* \overline{f}^*(p) = \pi_A^{-1}\left(\overline{f}^{-1}(p)\right) = \pi_A^{-1}\left\{a + \mathfrak{a} \in A/\mathfrak{a} \,|\, \overline{f}(a + \mathfrak{a}) = f(a) + \mathfrak{b} \in p\right\} = \{a \in A f(a) + \mathfrak{b} \in p\}$$

Going around the bottom of the rectangle, we obtain

$$f^* \pi_B^*(p) = f^{-1}\left(\pi_B^{-1}(p)\right) = f^{-1}\{b \in B \,|\, b + \mathfrak{b} \in p\} = \{a \in A \,|\, f(a) + \mathfrak{b} \in p\}$$

so the images are equal as subsets of A, so the are equal ideals in $\operatorname{spec} A$. □

Proposition 0.13 (Exercise 21(iv))**.** *Let $f : A \to B$ be a ring homomorphism. Let $\mathfrak{p} \in A$ be a prime ideal. Then $\operatorname{spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ is homeomorphic to $(f^*)^{-1}(\mathfrak{p})$.*

Proof. Applying Exercise 21(iii) to the ring homomorphism $S^{-1}f : A_\mathfrak{p} \to B_\mathfrak{p}$, we obtain the commutative diagram

$$\begin{array}{ccc}
\operatorname{spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) & \xrightarrow{\left(\overline{S^{-1}f}\right)^*} & \operatorname{spec}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \\
\cong \big\downarrow \pi_B^* & & \cong \big\downarrow \pi_A^* \\
V(\mathfrak{p}B_\mathfrak{p}) & & V(\mathfrak{p}A_\mathfrak{p}) \\
\big\downarrow & & \big\downarrow \\
\operatorname{spec} B_\mathfrak{p} & \xrightarrow{(S^{-1}f)^*} & \operatorname{spec} A_\mathfrak{p}
\end{array}$$

where π_B^*, π_A^* are the respective maps induced by the canonical projections $B_\mathfrak{p} \to B_\mathfrak{p}/\mathfrak{p}B\mathfrak{p}$ and $A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Using Exercise 21(ii) with $S = A \setminus \mathfrak{p}$, we extend our commutative diagram.

$$\begin{array}{ccc}
\operatorname{spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) & \xrightarrow{\left(\overline{S^{-1}f}\right)^*} & \operatorname{spec}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \\
\cong \big\downarrow \pi_B^* & & \cong \big\downarrow \pi_A^* \\
V(\mathfrak{p}B_\mathfrak{p}) & & V(\mathfrak{p}A_\mathfrak{p}) \\
\big\downarrow & & \big\downarrow \\
\operatorname{spec} B_\mathfrak{p} & \xrightarrow{(S^{-1}f)^*} & \operatorname{spec} A_\mathfrak{p} \\
\cong \big\downarrow \psi^* & & \cong \big\downarrow \phi^* \\
\operatorname{im} \psi^* & & \operatorname{im} \phi^* \\
\big\downarrow & & \big\downarrow \\
\operatorname{spec} B & \xrightarrow{f^*} & \operatorname{spec} A
\end{array}$$

If $\mathfrak{b}$ is an ideal of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$, then $\mathfrak{b}$ corresponds to an ideal of $B_\mathfrak{p}$ containing $\mathfrak{p}B_\mathfrak{p}$, which then corresponds to an ideal of B containing p^e. The image under f^* of such an element is then $\mathfrak{p}$. Conversely, if $\mathfrak{q} \in (f^*)^{-1}(\mathfrak{p}) \subset \operatorname{spec} B$, then

$$f^{-1}(\mathfrak{q}) = \mathfrak{p} \implies f(\mathfrak{p}) \subset \mathfrak{q} \implies \mathfrak{p}^e \subset \mathfrak{q} \implies \mathfrak{q} \cap (B \setminus \mathfrak{p}^e) = \emptyset$$

that is, $\mathfrak{q} \in \operatorname{im}\psi^*$, so $\mathfrak{q}$ corresponds to an ideal $\mathfrak{q}B_\mathfrak{p} \subset B_\mathfrak{p}$. Since $\mathfrak{p}^e \subset \mathfrak{q}$, $\mathfrak{q}B_\mathfrak{p}$ contains $\mathfrak{p}B_\mathfrak{p}$, thus $\mathfrak{q}B_\mathfrak{p}$ corresponds to an ideal of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$.

That is, the image of $\operatorname{spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ is precisely $(f^*)^{-1}(\mathfrak{p})$. Since the vertical maps in our commutative diagram are all injective, this gives the desired homeomorphism. $\square$